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ON GENERALIZATIONS OF THE PERRON-FROBENIUS THEOREM

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ABSTRACT

The Perron-Frobenius Theorem states that a matrix with nonnegative entries has at least one nonnegative eigenvalue of maximal absolute value and a corresponding eigenvector with nonnegative components. ~~In this paper~~ we discuss generalizations of this celebrated theorem that locate an eigenvalue of maximal absolute value and the components of a corresponding eigenvector within a certain angle of the complex plane depending on the angle which contains the entries of the matrix. A complete description of the 2×2 case as well as partial results for the general case are given.

1. Introduction and Statement of Results.

The Perron-Frobenius Theorem states that a matrix with nonnegative elements has at least one nonnegative eigenvalue of maximal absolute value. It seems natural to ask for generalizations that locate an eigenvalue of maximal absolute value within a certain angle of the complex plane depending on the angle which contains the elements of the matrix.

More precisely, let us consider the family $\underset{\sim}{A}_n(\alpha)$ of all $n \times n$ complex matrices $A = (a_{jk})$ whose entries are contained in a sector

$$\underset{\sim}{S}(\alpha) \equiv \{z: |\arg z| \leqq \alpha;\ 0 \leqq \alpha \leqq \pi \ \text{ fixed}\}.$$

For each $A \in \underset{\sim}{A}_n(\alpha)$ let $\beta = \beta(A)$ denote the minimal (nonnegative) angle for which the sector $\underset{\sim}{S}(\beta)$ contains an eigenvalue of A with maximal absolute value. Thus, defining

$$\beta_n(\alpha) = \sup_{A \in \underset{\sim}{A}_n(\alpha)} \beta(A),$$

we pose the following

PROBLEM 1. <u>Give</u> $\beta_n(\alpha)$ <u>as a function of</u> α <u>and</u> n.

The Perron-Frobenius Theorem states that

$$\beta_n(0) = 0, \quad n = 1,2,3,\dots .$$

Another simple observation is that for any $m < n$ and $A \in \underset{\sim}{A}_m(\alpha)$, we have $A \oplus 0_{n-m} \in \underset{\sim}{A}_n(\alpha)$, where 0_k is the $k \times k$ zero matrix. Thus,

(1.1) $\beta_n(\alpha)$ <u>is a nondecreasing function of</u> n.

Obviously,

$$\beta_1(\alpha) = \alpha; \tag{1.2}$$

so (1.1) and (1.2) yield

$$\beta_n(\alpha) \geqq \beta_1(\alpha) = \alpha, \quad n = 1,2,3,\ldots .$$

Since

$$\underset{\sim}{A}_n(\alpha) \subseteq \underset{\sim}{A}_n(\alpha') \quad \text{for} \quad \alpha' > \alpha,$$

we also find that

$\beta_n(\alpha)$ <u>is a nondecreasing function of</u> α.

In Section 2 we evaluate $\beta_2(\alpha)$ and obtain some partial results for arbitrary n, as stated in the following theorem:

THEOREM 1.

(i) <u>If</u> $n = 2$ <u>then</u>

$$\beta_2(\alpha) = \begin{cases} \alpha & \underline{\text{for}} \quad \alpha \leqq \frac{\pi}{4} \\ \alpha + \frac{\pi}{2} & \underline{\text{for}} \quad \frac{\pi}{4} < \alpha \leqq \frac{\pi}{2} \\ \pi & \underline{\text{for}} \quad \alpha > \frac{\pi}{2} . \end{cases}$$

(ii) <u>If</u> $n \geqq 3$ <u>is odd then</u>

$$\beta_n(\alpha) = \pi \quad \underline{\text{for}} \quad \alpha > \frac{\pi}{2n} .$$

(iii) <u>If</u> $n \geqq 4$ <u>is even then</u>

$$\beta_n(\alpha) \geqq \alpha + \frac{n-1}{n}\pi \quad \underline{\text{for}} \quad \frac{\pi}{2n} < \alpha \leqq \frac{\pi}{2n-2},$$

<u>and</u>

$$\beta_n(\alpha) = \pi \quad \underline{\text{for}} \quad \alpha > \frac{\pi}{2n-2}.$$

Note the discontinuity of $\beta_2(\alpha)$ at $\alpha = \pi/4$.

We also note that part (i) of the theorem implies that in general, A in $\underset{\sim}{A}_n(\alpha)$ does not have a square root in $\underset{\sim}{A}_n(\alpha/2)$.

Recall that the Perron-Frobenius Theorem not only proves that for matrices with nonnegative entries an eigenvalue of largest absolute value is positive, but also that a corresponding eigenvector can be chosen with nonnegative components.

With this in mind, for each $A \in \underset{\sim}{A}_n(\alpha)$ we denote $\gamma = \gamma(A)$ to be the minimal (nonnegative) angle such that the sector $\underset{\sim}{S}(\gamma)$ contains the components of an eigenvector corresponding to an eigenvalue of maximal absolute value which lies in $S(\beta(A))$. Hence, defining

$$\gamma_n(\alpha) = \sup_{A \in \underset{\sim}{A}_n(\alpha)} \gamma(A),$$

we pose:

PROBLEM 2. <u>Determine</u> $\gamma_n(\alpha)$ <u>as a function of</u> α <u>and</u> n.

The Perron-Frobenius Theorem tells us that

$$\gamma_n(0) = 0, \quad n = 1,2,3,\ldots,$$

and we obviously have

$$\gamma_1(\alpha) = 0.$$

Also, as for $\beta_n(\alpha)$, it is not hard to see that

$$\gamma_n(\alpha) \ \underline{\text{is a nondecreasing function of } n},$$

and

$$\gamma_n(\alpha) \ \underline{\text{is a nondecreasing function of } \alpha}.$$

Finally, since any eigenvector $(x_1,\ldots,x_n)' \in \underset{\sim}{C}^n$ (prime denoting the transpose) can be rotated so that its components are embedded in the sector $\underset{\sim}{S}(\pi-\pi/n)$ it follows that

$$\gamma_n(\alpha) \leq \frac{n-1}{n}\pi.$$

In analogy with the Perron-Frobenius Theorem it is perhaps natural to ask whether

$$\gamma_n(\alpha) = \beta_n(\alpha), \quad n = 2,3,4,\ldots,$$

or whether there is any nontrivial bound for $\gamma_n(\alpha)$ when $\alpha > 0$. This turns out to be incorrect, at least for all $n \geqq 4$, as stated in the following theorem which is proven in Section 2:

THEOREM 2.

(i) <u>If</u> $n = 2$ <u>then</u>

$$\gamma_2(\alpha) = \begin{cases} \alpha + \frac{\pi}{4} & \underline{\text{for}} \quad 0 < \alpha \leq \frac{\pi}{4} \\[2ex] \frac{\pi}{2} & \underline{\text{for}} \quad \alpha > \frac{\pi}{4}\,. \end{cases}$$

(ii) <u>If</u> $n = 3$ <u>then</u>

$$\gamma_3(\alpha) \geq 2\alpha + \frac{\pi}{2} \quad \underline{\text{for}} \quad 0 < \alpha \leq \frac{\pi}{12}$$

<u>and</u>

$$\gamma_3(\alpha) = \frac{2}{3}\pi \quad \underline{\text{for}} \quad \alpha > \frac{\pi}{12} .$$

(iii) <u>If</u> $n \geq 4$ <u>then</u>

$$\gamma_n(\alpha) = \frac{n-1}{n}\pi \quad \underline{\text{for}} \quad \alpha > 0. \tag{1.3}$$

For $n \geq 2$ note the discontinuity of $\gamma_n(\alpha)$ at $\alpha = 0$.

We also note that in fact, for $n = 2$, $\alpha \geq \pi/4$; $n = 3$, $\alpha \geq \pi/12$; and $n \geq 4$, $\alpha > 0$; $\gamma_n(\alpha)$ attains its maximal possible value.

Theorems 1 and 2 and the inequality in (1.3) can be summarized in the following two tables:

n	2	2	2	3	3	3
α	$0 < \alpha \leq \frac{\pi}{4}$	$\frac{\pi}{4} < \alpha \leq \frac{\pi}{2}$	$\alpha > \frac{\pi}{2}$	$0 < \alpha \leq \frac{\pi}{12}$	$\frac{\pi}{12} < \alpha \leq \frac{\pi}{6}$	$\alpha > \frac{\pi}{6}$
$\beta_n(\alpha)$	α	$\alpha + \frac{\pi}{2}$	π	$\geq \alpha$	$\geq \alpha$	π
$\gamma_n(\alpha)$	$\alpha + \frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\geq 2\alpha + \frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{2}{3}\pi$

Table 1. Cases $n = 2,3$

n	$n \geqq 4$ even	$n \geqq 4$ even	$n \geqq 4$ even	$n \geqq 5$ odd	$n \geqq 5$ odd
α	$0 < \alpha \leqq \frac{\pi}{2n}$	$\frac{\pi}{2n} < \alpha \leqq \frac{\pi}{2n-2}$	$\alpha > \frac{\pi}{2n-2}$	$0 < \alpha \leqq \frac{\pi}{2n}$	$\alpha > \frac{\pi}{2n}$
$\beta_n(\alpha)$	$\geqq \alpha$	$\geqq \alpha + \frac{n-1}{n}\pi$	π	$\geqq \alpha$	π
$\gamma_n(\alpha)$	$\frac{n-1}{n}\pi$	$\frac{n-1}{n}\pi$	$\frac{n-1}{n}\pi$	$\frac{n-1}{n}\pi$	$\frac{n-1}{n}\pi$

Table 2. The case $n \geqq 4$

We have not settled the following questions:

QUESTION 1. If $0 \leqq \alpha \leqq \pi/2n$ and $n \geqq 3$, is $\beta_n(\alpha) = \alpha$?

QUESTION 2. What is $\beta_n(\alpha)$ if $n \geqq 4$ is even and $\pi/2n < \alpha \leqq \pi/(2n-2)$?

QUESTION 3. If $0 < \alpha \leqq \pi/12$, what is $\gamma_3(\alpha)$?

2. Proofs.

Proof of Theorem 1. Since for all n and α we have

$$\alpha \leqq \beta_n(\alpha) \leqq \pi,$$

and since $\beta_n(\alpha)$ is a nondecreasing function of α, it remains to obtain the upper bounds

$$\beta_2(\alpha) \leqq \alpha, \quad \alpha \leqq \frac{\pi}{4}, \tag{2.1}$$

$$\beta_2(\alpha) \leqq \alpha + \frac{\pi}{2}, \ \frac{\pi}{4} < \alpha \leqq \frac{\pi}{2}, \tag{2.2}$$

and the lower bounds

$$\beta_n(\alpha) \geqq \alpha + \frac{n-1}{n}\pi, \quad n \geqq 2 \ \underline{\text{even}}, \quad \frac{\pi}{2\pi} < \alpha \leqq \frac{\pi}{2n-2}, \tag{2.3}$$

$$\beta_n(\alpha) \geqq \pi, \quad n \geqq 3 \ \underline{\text{odd}}, \quad \alpha > \frac{\pi}{2n}, \tag{2.4}$$

$$\beta_n(\alpha) \geqq \pi, \quad n \geqq 4 \ \underline{\text{even}}, \quad \alpha > \frac{\pi}{2n-2}. \tag{2.5}$$

We start with the upper bounds.

Since multiplication of a matrix by a constant multiplies the eigenvalues by the same constant, we may assume that the spectral radius of A, i.e., the maximal absolute value of the eigenvalues, is 1. Moreover, since the eigenvalues of the conjugate matrix $\bar{A} = (\bar{a}_{ij})$ are the complex conjugates of the eigenvalues of A, we can assume that $\lambda = e^{i\beta(A)}$ is an eigenvalue of A.

With these assumptions we first prove that

$$\beta(A) < \alpha + \pi/2 \quad \text{for} \quad A \in \underset{\sim}{A}_2(\alpha), \quad 0 \leqq \alpha \leqq \pi/2. \tag{2.6}$$

Let $A = (a_{jk}) \in \underset{\sim}{A}_2(\alpha)$ have eigenvalues $\lambda = e^{i\beta(A)}$ and λ'. Since

$$\lambda + \lambda' = \text{tr } A = a_{11} + a_{22} \in \underset{\sim}{S}(\alpha)$$

and since

$$\lambda' \in \underset{\sim}{U}$$

where $\underset{\sim}{U}$ is the unit disc centered at the origin, then

$$\lambda' \in \underset{\sim}{U} \cap (-\lambda + \underset{\sim}{S}(\alpha)).$$

Thus,

$$\lambda + \lambda' \in (\lambda + \underset{\sim}{U}) \cap \underset{\sim}{S}(\alpha) \equiv \underset{\sim}{T} \tag{2.7}$$

so that $\underset{\sim}{T}$ is the intersection of the unit disc centered at λ and the sector $\underset{\sim}{S}(\alpha)$. Therefore, if

$$\beta(A) \geqq \alpha + \pi/2,$$

then $\underset{\sim}{T} = \{0\}$, and by (2.7), $\lambda' = -\lambda$. Hence, $|\lambda'| = 1$ with $|\arg \lambda'| < \arg \lambda = \beta(A)$ in contradiction to the definition of $\beta(A)$. Thus we have (2.6), and (2.2) follows.

Now, fix α with

$$0 \leqq \alpha \leqq \pi/4,$$

and let us prove (2.1).

Suppose that for some $A = (a_{jk}) \in \underset{\sim}{A}_2(\alpha)$ with eigenvalues $\lambda = e^{i\beta(A)}$ and λ', $|\lambda'| \leq 1$, we have

$$\alpha < \beta(A) < \alpha + \pi/2. \tag{2.8}$$

Since $\beta(A) < \alpha + \pi/2$, then $\underset{\sim}{T} \neq \{0\}$. Further, since $\beta(A) > \alpha$, then if $\max_j \arg a_{jj} < \alpha$, there exists a positive number r so that

$$\max_{j=1,2} \arg(a_{jj} \qquad \lambda) = \alpha.$$

We can thus replace A by the matrix

$$A_0 = \frac{1}{1+r}(A + r\lambda I) \in \underset{\sim}{A}_2(\alpha)$$

whose eigenvalues are λ and $\lambda'' = (\lambda' + r\lambda)/(1 + r)$. Hence, $|\lambda''| < 1$, unless $\lambda' = \lambda = \frac{1}{2}\operatorname{tr} A \in \underset{\sim}{S}(A)$ which contradicts the assumption that $\beta(A) > \alpha$. So, we may use A_0 instead of A, drop the subscript, and assume without loss of generality that

$$\arg a_{11} = \alpha. \tag{2.9}$$

By (2.7) we have

$$a_{11} + a_{22} = \operatorname{tr} A \in \underset{\sim}{T}; \tag{2.10}$$

so

$$a_{22} \in -a_{11} + \underset{\sim}{T}.$$

Also, by (2.9), (2.10), and since $a_{22} \in \underset{\sim}{S}(\alpha)$, we have $a_{11} \in \underset{\sim}{T}$; so if we denote by a'_{11} the point symmetric to a_{11} in $\underset{\sim}{T}$, then (see Figure 1)

$$a'_{11} = 2e^{i\alpha}\cos(\beta - \alpha) - a_{11} \in -a_{11} + \underset{\sim}{T}.$$

We may not have

$$a_{22} = a'_{11}.$$

For in that case

$$\lambda' = \mathrm{tr}\, A - \lambda = a_{11} + a'_{11} - \lambda = e^{i(2\alpha-\beta)};$$

hence $|\lambda'| = 1$ and by (2.8), $|\arg \lambda'| < |\arg \lambda|$ which contradicts the definition of β. Therefore,

$$a_{22} \neq a'_{11};$$

so

$$\arg(\lambda - a_{22}) < \arg(\lambda - a'_{11}),$$

and

$$\text{(2.11)} \qquad \arg(\lambda - a_{.1})(\lambda - a_{22}) = \arg(\lambda - a_{11}) + \arg(\lambda - a_{22})$$

$$< \arg(\lambda - a_{11}) + \arg(\lambda - a'_{11}) = (\theta + \alpha) + (\pi - \theta + \alpha) = \pi + 2\alpha.$$

Since $A \in \underset{\sim}{A}_n(\alpha)$ we also have

$$\text{(2.12)} \quad \arg(\lambda - a_{11})(\lambda - a_{22}) = \arg(\lambda - a_{11}) + \arg(\lambda - a_{22}) \geqq 2\beta > 2\alpha,$$

and

$$\text{(2.13)} \qquad -2\alpha \leqq \arg a_{12}a_{21} \leqq 2\alpha.$$

But now, since $0 \leqq \alpha \leqq \pi/4$, equations (2.11)-(2.13) contradict the characteristic equation of A,

$$(\lambda - a_{11})(\lambda - a_{22}) = a_{12}a_{21}.$$

Thus, assumption (2.8) fails; so in view of (2.6) we must have

$$\beta(A) \leq \alpha, \quad 0 \leq \alpha < \pi/4,$$

and (2.1) follows.

Figure 1.

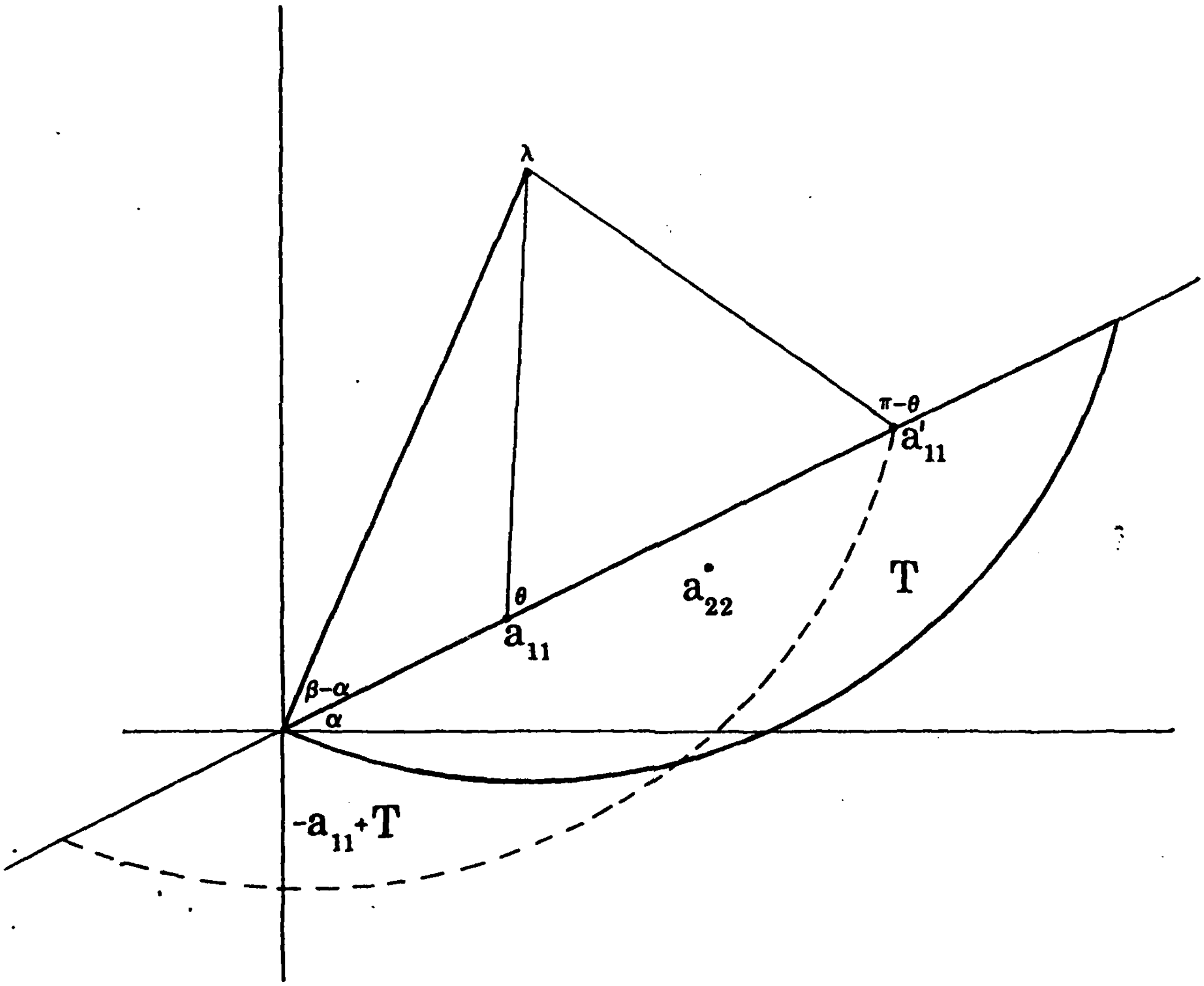

Figure 1

The main tools in obtaining the lower bounds in (2.3)-(2.5) are the $n \times n$ circulants

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & & a_{n-3} & a_{n-2} \\ \vdots & \vdots & & & \\ a_1 & a_2 & & a_{n-1} & a_0 \end{pmatrix} \tag{2.14}$$

whose eigenvalues are

$$\lambda_j = a_0 + a_1\omega_j + a_2\omega_j^2 + \ldots + a_{n-1}\omega_j^{n-1}, \quad j = 1,\ldots,n,$$

where

$$\omega_j = e^{2\pi ij/n}, \quad j = 1,\ldots,n,$$

are the n-th roots of unity.

Assume now that

$$\alpha > \pi/2n,$$

and set the a_j in (2.14) to zero except for the entries

$$a_m = e^{i\alpha}, \quad a_{m+1} = re^{i(\alpha-\pi/n-\varepsilon)},$$

$$r > 0, \quad 0 < \varepsilon < \pi/2n, \quad 0 \le m \le n-2,$$

where ε is chosen sufficiently small so that $A \in \underset{\sim}{A}_n(\alpha)$ and m will be determined later. The eigenvalues are

$$\lambda_j = (a_m + \omega_j a_{m+1})\omega_j^m, \quad j = 1,\ldots,n,$$

and we see that their maximal absolute value is obtained for the j that minimizes

$$|\arg a_m - \arg(\omega_j a_{m+1})| = |\arg a_m - \arg \omega_j - \arg a_{m+1}| = \left|\frac{\pi}{n} - \frac{2\pi j}{n} + \varepsilon\right|.$$

Since $0 < \varepsilon < \pi/2n$, this occurs at $j = 1$; thus,

$$|\lambda_1| > |\lambda_j| , \quad j = 2,\ldots,n;$$

and consequently

$$\beta(A) = |\arg \lambda_1|.$$

Moreover, since the argument of $a_m + \omega_1 a_{m+1}$ goes from $\arg a_m = \alpha$ to $\arg \omega_1 a_{m+1} = \alpha + \pi/n - \varepsilon$ as r goes from zero to infinity, then

$$\arg \lambda_1 = \arg(a_m + \omega_1 a_{m+1})\omega_1^m = \arg(a_m + \omega_1 a_{m+1}) + 2\pi m/n$$

runs through the interval

$$\text{(2.15)} \qquad \underset{\sim}{I}_m = \left(\alpha + \frac{2\pi m}{n} , \alpha + \frac{2m+1}{n}\pi - \varepsilon\right).$$

Now, if n is odd and

$$\pi/2n < \alpha < \pi/n,$$

we pick

$$m = \frac{n-1}{2} ;$$

so the interval $\underset{\sim}{I}_m$ in (2.15) contains π. Hence, there exists a circulant $A \in \underset{\sim}{A}_n(\alpha)$ such that

$$\beta(A) = |\arg \lambda_1| = \pi.$$

Therefore,

$$\beta_n(\alpha) \geqq \pi, \quad n \geqq 3 \quad \underline{\text{odd}}, \quad \frac{\pi}{2n} < \alpha < \frac{\pi}{n},$$

and by the monotonicity of $\beta_n(\alpha)$ in (1.4) we get (2.4).

If n is even and

$$\pi/2n < \alpha \leqq \pi/(2n-2),$$

we choose

$$m = \frac{n-2}{2},$$

so that the interval $\underset{\sim}{I}_m$ contains the point $\alpha + \pi - \pi/n - 2\varepsilon$. Thus, we can find a circulant $A \in \underset{\sim}{A}_n(\alpha)$ with

$$\beta(A) = |\arg \lambda_1| = \alpha + \pi - \pi/n - 2\varepsilon;$$

and since ε was arbitrarily small, we have

$$\beta_n(\alpha) \geqq \alpha + \frac{n-1}{n}\pi, \quad n \geqq 2 \quad \underline{\text{even}}, \quad \frac{n}{2n} < \alpha \leqq \frac{n}{2n-2},$$

so (2.3) is established.

Finally, if n is even and

$$\alpha > \pi/(2n-2),$$

then

$$\alpha > \pi/2\ell$$

where $\ell \equiv n-1$ is odd. Thus, by (2.4),

$$\beta_\ell(\alpha) \geqq \pi;$$

and since $\beta_n(\alpha)$ is a nondecreasing function of n, we obtain (2.5). □

Proof of Theorem 2. Since for all n and α we have

$$\gamma_n(\alpha) \leqq \frac{n-1}{n}\pi$$

and since $\gamma_n(\alpha)$ is a nondecreasing function of α, it suffices to obtain the upper bound

$$\gamma_2(\alpha) \leqq \alpha + \frac{\pi}{4}, \quad 0 < \alpha \leqq \frac{\pi}{4}, \tag{2.16}$$

and the lower bounds

$$\gamma_2(\alpha) \geqq \alpha + \frac{\pi}{4}, \quad 0 < \alpha \leqq \frac{\pi}{4}, \tag{2.17}$$

$$\gamma_3(\alpha) \geqq 2\alpha + \frac{\pi}{2}, \quad 0 < \alpha \leqq \frac{\pi}{12}, \tag{2.18}$$

$$\gamma_n(\alpha) \gtrsim \frac{n-1}{n}\pi, \quad n \geqq 4, \quad \alpha > 0. \tag{2.19}$$

Starting with the upper bound in (2.16), we let

$$\alpha \leqq \pi/4,$$

and take any $A = (a_{jk}) \in \underset{\sim}{A}_2(\alpha)$. As in the previous proof, we may assume without loss of generality that the eigenvalues of A are λ and λ' with

$$\lambda = e^{i\beta(A)}, \quad |\lambda'| \leq 1.$$

Now let $(x_1,x_2)'$ be an eigenvector corresponding to λ. If one of the components vanishes, the other may be taken to be 1; so in this case $\gamma(A) = 0$. If the eigenvector does not have a zero coordinate we may assume that it is of the form $(1,x)'$ where $x \neq 0$, thus

$$\gamma(A) = \frac{1}{2}|\arg x|. \tag{2.20}$$

We have

$$a_{11} + a_{12}x = \lambda,$$
$$a_{21} + a_{22}x = \lambda x,$$

and therefore

$$x = \frac{\lambda - a_{11}}{a_{12}} = \frac{a_{21}}{\lambda - a_{22}} . \tag{2.21}$$

Since $\alpha \leq \pi/4$, then by part (i) of Theorem 1 we have $\beta(A) \leq \alpha$ so the interval connecting the origin and the point λ lies in $\underset{\sim}{S}(\alpha)$, and the line Γ through λ perpendicular to the interval $[0,\lambda]$ (see Figure 2) must intersect the positive real axis. Now, if a_{11} and a_{22} are located on the right of Γ then

$$|a_{11} + a_{22}| > 2\,|\lambda| = 2,$$

in contradiction to the fact that

$$|a_{11} + a_{22}| = |\lambda + \lambda'| \leq |\lambda| + |\lambda'| \leq 2.$$

Thus, say, a_{11} is on the (closed) left side of Γ, and

$$|\arg(\lambda - a_{11})| \leqq |\arg \lambda| + \frac{\pi}{2} = \beta + \frac{\pi}{2} .$$

So finally, by (2.20) and (2.21), and since $\beta \leqq \alpha$, we have

$$\gamma(A) = \frac{1}{2}|\arg x| = \frac{1}{2}\left|\arg \frac{\lambda - a_{11}}{a_{12}}\right|$$

$$\leqq \frac{1}{2}|\arg(\lambda - a_{11})| + \frac{1}{2}|\arg a_{12}| \leqq \frac{1}{2}\left(\beta + \frac{\pi}{2} + \alpha\right) \leqq \alpha + \frac{\pi}{4} ,$$

and (2.16) follows.

Figure 2

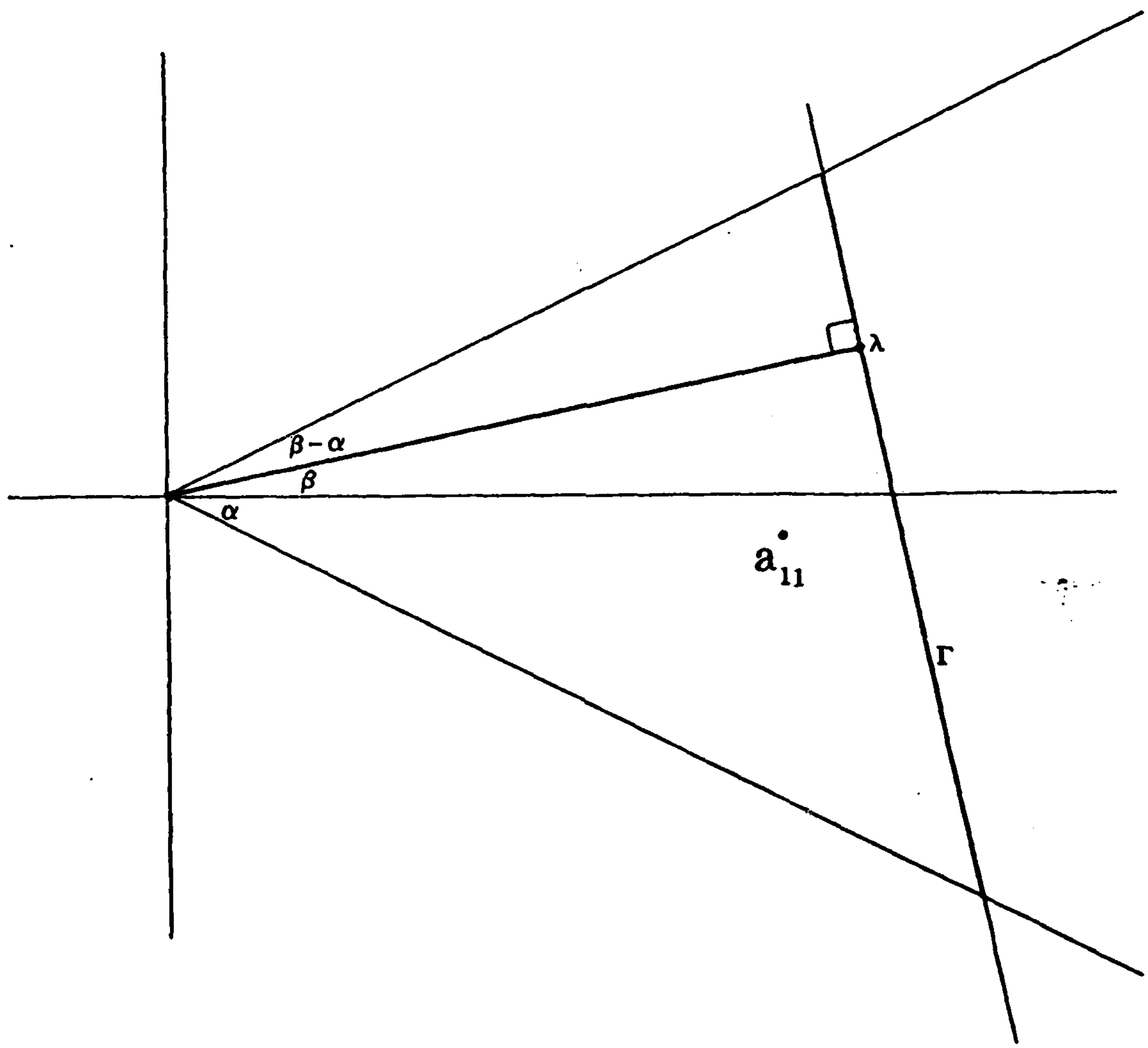

Figure 2

For the lower bounds, our tools are $n \times n$ lower triangular matrices of the form

$$A = \begin{pmatrix} \lambda & & & & \bigcirc \\ a & \lambda' & & & \\ & a & \lambda' & & \\ & & \ddots & \ddots & \\ \bigcirc & & & a & \lambda' \end{pmatrix}, \; |\lambda'| < |\lambda|,$$

where evidently, the eigenvector belonging to λ is given, up to an arbitrary factor, by

(2.22) $\qquad (1, x, x^2, \ldots, x^{n-1})'$ with $x = \dfrac{a}{\lambda - \lambda'}$.

With these matrices, we now make suitable choices for the quantities λ, λ' and a for different values of n.

First, for $n = 2$ and

$$0 < \alpha \leq \pi/4,$$

set

$$\lambda = e^{i\alpha}, \quad \lambda' = \lambda - \delta e^{i(\alpha+\pi/2-\varepsilon)}, \quad a = \delta\bar{\lambda},$$

where $\varepsilon > 0$ is small and then $\delta > 0$ is chosen so small that $|\lambda'| < 1$ and $A \in \underset{\sim}{A}_n(\alpha)$. The vector in (2.22) is now

$$(1,x)', \quad x = e^{-i(2\alpha+\pi/2-\varepsilon)};$$

thus

$$\gamma(A) = \frac{1}{2}|\arg x| = \alpha + \frac{\pi}{4} - \frac{\varepsilon}{2} ,$$

and since ε is arbitrarily small we obtain (2.17).

For $n = 3$ and

$$0 < \alpha \leqq \pi/12,$$

we make the same choice of λ, λ', and a as for $n = 2$, so the eigenvector in (2.22) is

$$(1, x, x^2)', \quad x = e^{-i(2\alpha+\pi/2-\delta)}.$$

Therefore,

$$\gamma(A) = \frac{1}{2}|\arg x^2| = 2\alpha + \pi/2 - \varepsilon,$$

and again, since ε is arbitrarily small, we obtain (2.18).

For $n = 4$ and $\alpha > 0$ we set

$$\lambda = e^{-i\varepsilon}, \quad \lambda' = \cos\varepsilon, \quad a = \sin\varepsilon, \quad 0 < \varepsilon < \alpha.$$

The corresponding eigenvector in (2.22) is $(1, i, -1, -i)'$; so

$$\gamma(A) = \frac{3}{4}\pi$$

and (2.19) holds for $n = 4$.

Finally, if $n > 4$ and $\alpha > 0$, our choice is

$$\lambda = 1, \quad \lambda' = 1 - \varepsilon\bar{\omega}_1, \quad a = \varepsilon, \quad \varepsilon > 0, \quad \omega_1 = e^{2\pi i/n},$$

where again, if ε is sufficiently small then $A \in \underset{\sim}{A}_n(\alpha)$ and $|\lambda'| < 1$.

The eigenvector in (2.22) is now

$$(1, \omega, \omega_1^2, \ldots, \omega_1^{n-1})';$$

so

$$\gamma(A) = \frac{n-1}{n}\pi ,$$

and the proof of (2.20) is complete. □

Note that the lower bounds in (2.17)-(2.19) were obtained for matrices in $\underset{\sim}{A}_n(\alpha)$ whose eigenvalues are <u>all</u> contained in $\underset{\sim}{S}(\alpha)$. In fact, for $n \geq 4$ we found matrices A such that $\gamma(A)$ obtained the maximal possible value, while all the eigenvalues were arbitrarily close to 1.

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The Perron-Frobenius Theorem states that a matrix with nonnegative entries has at least one nonnegative eigenvalue of maximal absolute value and a corresponding eigenvector with nonnegative components. In this paper the authors discuss generalizations of this celebrated theorem that locate an eigenvalue of maximal absolute value and the components of a corresponding eigenvector within a certain angle of the complex plane depending on the angle which contains the entries of the matrix. A complete description of the 2x2 case as well as partial results for the general case are given.

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